On Dispersive Decay for 3D Discrete Schrödinger and Klein-Gordon Equations

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Abstract

We derive the long-time decay in weighted norms for solutions of the discrete 3D Schrödinger and Klein-Gordon equations.

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1 Introduction

In this paper we establish the long-time behaviour of the solutions to the discrete three dimensional Schrödinger and Klein-Gordon equations. We extend a general strategy introduced by Vainberg [16], Jensen-Kato [7] and Murata [12] which concerns the wave, Klein-Gordon and Schrödinger equations, to the discrete case. Namely, we establish the smoothness in the continuous spectrum and the Puiseux expansion for the resolvent of the generator. Then the long-time asymptotics can be obtained by means of the (inverse) Fourier-Laplace transform.

We restrict ourselves to a "nonsingular case", in the sense of [12], where the truncated resolvent is bounded at the ends of the continuous spectrum. This holds for a generic potential and allows us to get the decay of order $\sim t^{-3/2}$ which is desirable for applications to scattering problems.

First we consider the 3D discrete version of the Schrödinger equation,

$$\left\{\begin{array}{l} i\dot{\psi}(x,t) = H\psi(x,t) := (-\Delta + V(x))\,\psi(x,t) \\ \\ \psi\big|_{t=0} = \psi_0 \end{array}\right| \quad x\in\mathbb{Z}^3, \quad t\in\mathbb{R}, \qquad (1.1)$$

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where Δ stands for the difference Laplacian in $\mathbb{Z}^3$, defined by

$$\Delta\psi(x) = \sum_{|y-x|=1} \psi(y) - 6\psi(x), \quad x \in \mathbb{Z}^3,$$

for functions $\psi : \mathbb{Z}^3 \to \mathbb{C}$.

Definition 1.1. *Denote by $\mathcal{V}$ the set of real valued functions on the lattice $\mathbb{Z}^3$ with finite supports.*

For the potential V, we assume that $V \in \mathcal{V}$. The Fourier-Laplace transform

$$\tilde{\psi}(x,\omega) = \int_0^\infty e^{i\omega t}\psi(x,t)\, dt, \quad \operatorname{Im}\omega > 0 \tag{1.2}$$

reduces (1.1) to corresponding stationary equation

$$(H-\omega)\tilde{\psi}(\omega) = -i\psi_0, \quad \operatorname{Im}\omega > 0. \tag{1.3}$$

Therefore,

$$\tilde{\psi}(\cdot,\omega) = -iR(\omega)\psi_0, \tag{1.4}$$

where $R(\omega) = (H-\omega)^{-1}$ is the resolvent of the Schrödinger operator H.

We are going to use the functional spaces which are the discrete version of the Agmon spaces [1]. These spaces are the weighted Hilbert spaces $l^2_\sigma = l^2_\sigma(\mathbb{Z}^3)$ with the norm

$$\|u\|_{l^2_\sigma} = \left\|(1+x^2)^{\sigma/2}u\right\|_{l^2}, \quad \sigma \in \mathbb{R}.$$

Let us denote

$$B(\sigma,\, \sigma') = \mathcal{L}(l^2_\sigma,\, l^2_{\sigma'}), \quad \mathbf{B}(\sigma,\, \sigma') = \mathcal{L}(l^2_\sigma \oplus l^2_\sigma,\, l^2_{\sigma'} \oplus l^2_{\sigma'})$$

the spaces of bounded linear operators from l^2_σ to $l^2_{\sigma'}$ and from $l^2_\sigma \oplus l^2_\sigma$ to $l^2_{\sigma'} \oplus l^2_{\sigma'}$, respectively. We write $K = \operatorname{Op}(K(x,y))$ for the operator with kernel $K(x,y)$, i.e.,

$$(Ku)(x) = \sum_{y\in\mathbb{Z}^3} K(x,y)u(y), \quad x \in \mathbb{Z}^3.$$

Note that the continuous spectrum of the operator H coincides with the interval $[0,12]$, and the resolvent has singularities at points $\omega_k = 4k$, $k = 0,1,2,3$ (see [4]). Our main results are as follows. For "a generic potential" $V \in \mathcal{V}$ (see Definition 3.3), we obtain the Puiseux expansion for the resolvent at the singular spectral points ω_k:

$$R(\omega_k + \omega) = R_{k0} + \mathcal{O}(\sqrt{\omega}), \quad \omega \to 0, \tag{1.5}$$

in the norm $B(\sigma; -\sigma)$ with $\sigma > 7/2$. Then for initial data $\psi_0 \in l^2_\sigma$ with $\sigma > 7/2$ we obtain the following long-time asymptotics:

$$\psi(\cdot,t) = \sum_{j=1}^n C_j e^{-it\mu_j}u_j + U_0(t)\phi_\pm + r_\pm(t), \quad t \to \pm\infty. \tag{1.6}$$

Here u_j are the eigenfunctions of operator H, corresponding to the discrete eigenvalues μ_j, $U_0(t)$ -dynamical group of free Schrödinger equation, $\phi_\pm \in l^2$ - asymptotic scattering state, and $\|r_\pm(t)\|_{l^2} = \mathcal{O}(|t|^{-1/2})$.

For the proof, we develop the arguments similar to [7], [12]. Lemma 10.2 of Jensen-Kato [7] plays the crucial role in verifying the decay (1.6)

We obtain the asymptotics (1.5) from the invertibility of operator $1 + A_{k0}V$, where A_{k0} are the coefficients with ω^0 in the expansion of free resolvent (see. (2.12), (2.20)). Note that operator $1 + A_{k0}V$ is invertible for "a generic potential". Asymptotics (1.5) implies the boundedness of the truncated resolvent at the singular points of the continuous spectrum. In the other words, this points are neither eigenvalue nor resonance for the operator H. Using the methods of [12] one can easy proved that the boundedness of the truncated resolvent is equivalent to invertibility of the operator $1 + A_{k0}V$.

We also obtain similar results for the discrete Klein-Gordon equation:

$$\left\{ \begin{array}{l} \ddot{\psi}(x,t) = (\Delta - m^2 - V(x))\,\psi(x,t) \\ \\ \psi\big|_{t=0} = \psi_0,\ \dot{\psi}\big|_{t=0} = \pi_0 \end{array} \right| \quad x \in \mathbb{Z}^3, \quad t \in \mathbb{R}. \tag{1.7}$$

Set $\mathbf{\Psi}(t) \equiv \big(\psi(\cdot,t), \dot{\psi}(\cdot,t)\big)$, $\mathbf{\Psi}_0 \equiv \big(\psi_0, \pi_0\big)$. Then (1.7) takes the form

$$i\dot{\mathbf{\Psi}}(t) = \mathbf{H}\mathbf{\Psi}(t), \quad t \in \mathbb{R}; \quad \mathbf{\Psi}(0) = \mathbf{\Psi}_0, \tag{1.8}$$

where

$$\mathbf{H} = \left(\begin{array}{cc} 0 & i \\ i(\Delta - m^2 - V) & 0 \end{array} \right)$$

The resolvent $\mathbf{R}(\omega) = (\mathbf{H} - \omega)^{-1}$ of the operator $\mathbf{H}$ can be expressed in terms of the resolvent $R(\omega)$, and this expression yields the corresponding properties of $\mathbf{R}(\omega)$. In particular, we obtain the asymptotic expansion of type (1.5) for $\mathbf{R}(\omega)$, and also the long-time asymptotics of type (1.6) for the solution of (1.8).

Let us comment on previous results in this direction. Eskina [3], and Shaban and Vainberg [14] considered the difference Schrödinger equation in dimension $n \geq 1$. They proved the limiting absorption principle for matrix elements of the resolvent and applied it to the Sommerfeld radiation condition. However, [3, 14] do not concern the asymptotic expansion of $R(\omega)$ and the long-time asymptotics of type (1.6) in the operator norms.

The asymptotic expansion of the matrix element of the resolvent $R(\omega)$ at the singular points ω_k was obtained by Islami and Vainberg [4]. They used this expansion to prove the long time asymptotics for the solutions of the Cauchy problem for the difference wave equation. The main feature which differs the present paper from [4] is that here all asymptotic expansions hold in the weighted functional spaces, not on compacts as in [4]. In fact, the asymptotic expansion of the resolvent (1.5) in the norm of $B(\sigma, -\sigma)$ was the main technical challenge in this paper. Additional difference is that the long time asymptotics here is obtained for the Schrödinger and Klein-Gordon equations, contrary to the wave equation in [4].

The asymptotic expansion of the resolvent and the long time asymptotics (1.6) for hyperbolic PDEs in $\mathbb{R}^n$ (continuous case) were obtained earlier in [11], [15], [16], [17], and for the Schrödinger equation in [7], [6], [5], [12]; also see [13] for an up-to-date review and many references concerning dispersive properties of solutions to the continuous Schrödinger equation in various norms.

The results of present paper extend the results of [9] and [10] from difference 1D and 2D equations to difference 3D equations.

The paper is organised as follows. In Section 2 we derive the asymptotic expansion of the free resolvent. The limiting absorption principle and the Puiseux expansion of the perturbed resolvent is proved in Section 3. In Section 4 we prove the long-time asymptotics (1.6). In Section 6 we extend the results to the discrete Klein-Gordon equation.

2 The free resolvent

We start with an investigation of the unperturbed problem for the equation (1.1) with $V(x) = 0$. The discrete Fourier transform of $u : \mathbb{Z}^3 \to \mathbb{C}$ is defined by the formula

$$\widehat{u}(\theta) = \sum_{x \in \mathbb{Z}^3} u(x) e^{i\theta x}, \ \theta \in T^3 := \mathbb{R}^3 / 2\pi\mathbb{Z}^3,$$

After taking the Fourier transform, the operator $H_0 = -\Delta$ becomes the operator of multiplication by $\phi(\theta) := 6 - 2\sum\limits_{j=1}^{3} \cos\theta_j = 4\sum\limits_{j=1}^{3} \sin^2\frac{\theta_j}{2}$:

$$-\widehat{\Delta u}(\theta) = \phi(\theta)\widehat{u}(\theta), \quad \theta \in T^3. \tag{2.1}$$

Thus, the operator H_0 is selfadjoint and its spectrum coincides with the range of the function ϕ, that is $\mathrm{Spec} H_0 = \Sigma := [0, 12]$. Denote by $R_0(\omega) = (H_0 - \omega)^{-1}$ the resolvent of the difference Laplacian. Then the kernel of the resolvent $R_0(\omega)$ reads

$$R_0(\omega, x - y) = \frac{1}{8\pi^3} \int\limits_{T^3} \frac{e^{-i\theta(x-y)}}{\phi(\theta) - \omega}\, d\theta, \ \omega \in \mathbb{C} \setminus \Sigma. \tag{2.2}$$

Lemma 2.1. *The free resolvent $R_0(\omega)$ is an analytic function of $\omega \in \mathbb{C} \setminus \Sigma$ with the values in $\mathcal{B}(\sigma, \sigma')$ for any $\sigma, \sigma' \in \mathbb{R}$.*

Proof. For a fixed $\omega \in \mathbb{C} \setminus \Sigma$, we have $\phi(\theta) - \omega \neq 0$ for $\theta \in T^3$. Therefore, $\phi(\theta + i\xi) - \omega \neq 0$ for $\theta \in T^3$, $\xi \in \mathbb{R}^3$, if $\xi \neq 0$ is sufficiently small. Hence, the function $1/(\phi(\theta) - \omega)$ admits analytic continuation into a complex neighbourhood of the torus of type $\{\theta + i\xi : \theta \in T^3, \ \xi \in \mathbb{R}^3 : |\xi| < \delta(\omega)\}$ with an $\delta(\omega) > 0$. Therefore the Paley-Wiener arguments imply that

$$R_0(\omega, x - y) \leq C(\delta) e^{-\delta|x-y|}$$

for any $\delta < \delta(\omega)$. Hence, $R_0(\omega)$ is the Hilbert-Schmidt operator in the space $\mathcal{B}(\sigma, \sigma')$. $\square$

2.1 Limiting absorption principle

Now we are interested in the traces of the analytic function $R_0(\omega)$ at the cut Σ. We can write

$$R_0(\omega \pm i\varepsilon, z) = F^{-1}_{\theta \to z} \frac{1}{\phi(\theta) - \omega \mp i\varepsilon} = \frac{1}{8\pi^3} \int\limits_{T^3} \frac{e^{-i\theta z}}{\phi(\theta) - \omega \mp i\varepsilon}\, d\theta, \quad z \in \mathbb{Z}^3 \tag{2.3}$$

with $\varepsilon > 0$. Note that the limiting distribution $\dfrac{1}{\phi(\theta)-\omega\mp i0}$ is well defined if ω is not a critical value of the function $\phi(\theta)$, i.e. the level line $\phi(\theta)=\omega$ does not contain the critical points with $\nabla\phi(\theta)=0$. The critical points $\theta=(\theta_1,\theta_2,\theta_3)$ can be easily calculated: $\theta_i=k_i\pi$, $k_i\in\mathbb{Z}$. Therefore, the critical values are $\omega=0,4,8,12$. For the free resolvent the following limiting absorption principle holds

Proposition 2.2. *For $\sigma>3/2$ the following limits exist as $\varepsilon\to 0+$:*

$$R_0(\omega\pm i\varepsilon)\ \xrightarrow{\mathcal{B}(\sigma,-\sigma)}\ R_0(\omega\pm i0),\quad \omega\in\Sigma\setminus\{0,4,8,12\}. \tag{2.4}$$

Proof. First we prove the convergence (2.4) for any fixed z, follow [3]. Let $\chi_j(\theta)$, $j=1,...,l$ are the sufficient small partition of unity on the torus T^3. Then

$$R_0(\omega\pm i\varepsilon,z)=\sum_{j=1}^{l}\frac{1}{8\pi^3}\int\limits_{D_j}\frac{\chi_j(\theta)e^{-i\theta z}}{\phi(\theta)-\omega\mp i\varepsilon}\,d\theta=\sum_{j=1}^{l}R_{0,j}(\omega\pm i\varepsilon,z), \tag{2.5}$$

where $D_j=\operatorname{supp}\chi_j$. If $\{\phi(\theta)=\omega\}\cap D_j=\emptyset$, then the function $R_{0,j}(\omega\pm i\varepsilon,z)$ is continuous for $\varepsilon\ge 0$ and

$$|R_{0,j}(\omega\pm i\varepsilon,z)|<C_j<\infty,\quad z\in Z^3,\quad \varepsilon\ge 0. \tag{2.6}$$

Now let $S_j=\{\phi(\theta)=\omega\}\cap D_j$. Then $\forall\theta\in D_j$

$$\theta=s+tn(s),$$

where $s\in S_j$, and $n(s)$ is the external normal vector to S_j at the point s of unit length. Let us introduce the new variables (s,t). Then

$$R_{0,j}(\omega\pm i\varepsilon,z)=\frac{1}{8\pi^3}\int\limits_{S_j}e^{-isz}ds\int\limits_{-a(s)}^{b(s)}\frac{\chi_j(s+tn(s))e^{-itn(s)z}J(s,t)}{t\psi(s+tn(s))\mp i\varepsilon}\,dt. \tag{2.7}$$

where $J(s,t)$ is the Jacobian, ψ is the smooth function, and $a(s)$, $b(s)>0$. Note that $J(s,t)|_{t=0}=1$, and $\psi(s+tn(s))|_{t=0}=|\nabla\phi(s)|$. We will prove the following lemma:

Lemma 2.3. *Let $\varphi(t,z)$ is the smooth function satisfies*

$$|\varphi(t,z)|\le C,\quad |\partial_t\varphi(t,z)|\le C|z|,\quad t\in[-\delta,\delta],\ z\in\mathbb{Z}^3, \tag{2.8}$$

and let $\psi(t)$ is the smooth function such that $\psi(t)\ne 0$ if $t\in[-\delta,\delta]$. Consider $F(\pm\varepsilon,z)=\int\limits_{-\delta}^{\delta}\dfrac{\varphi(t,z)}{t\psi(t)\mp i\varepsilon}dt$. Then

$$F(\pm\varepsilon,z)\to F(\pm 0,z),\quad ,\varepsilon\to 0+,\quad z\in\mathbb{Z}^3, \tag{2.9}$$

and

$$\sup_{\varepsilon\in(0,1]}|F(\pm\varepsilon,z)|\le C\Big(\ln(1+|z|)+1\Big),\quad z\in\mathbb{Z}^3. \tag{2.10}$$

Proof. Let us rewrite $F(\pm\varepsilon,z)$ as

$$F(\pm\varepsilon,z)=\varphi(0,z)\int\limits_{-\delta}^{\delta}\frac{dt}{t\psi(0)\mp i\varepsilon}-\varphi(0,z)\int\limits_{-\delta}^{\delta}\frac{(\psi(t)-\psi(0))tdt}{(t\psi(t)\mp i\varepsilon)(t\psi(0)\mp i\varepsilon)}+\int\limits_{-\delta}^{\delta}\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)\mp i\varepsilon}dt.$$

Then,

$$F(\pm\varepsilon,z)\to F(\pm 0,z)=\pm i\pi\frac{\varphi(0,z)}{\psi(0)}-\frac{\varphi(0,z)}{\psi(0)}\int\limits_{-\delta}^{\delta}\frac{\psi(t)-\psi(0)}{t\psi(t)}dt+\int\limits_{-\delta}^{\delta}\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)}dt \quad (2.11)$$

as $\varepsilon\to 0+$. By (2.8) the first and the second summand in RHS of (2.11) can be estimated by the constant which does not depend on $|z|\in\mathbb{Z}^3$. Let us estimate the third summand in RHS of (2.11). For $|z|<1/\delta$ this summand also can be estimated by the constant. For $|z|>1/\delta$ we obtain by (2.8) that

$$\int\limits_{-\delta}^{\delta}\Big|\frac{\varphi(t,z)-\varphi(0,z)}{t\psi(t)}\Big|dt=\int\limits_{|t|<1/|z|}...+\int\limits_{1/|z|<|t|<\delta}...\le\frac{1}{|z|}C|z|+C\ln|z|\le C\ln|z|.$$

Lemma is proved. ◻

Lemma 2.3 implies that

$$R_{0j}(\omega\pm i\varepsilon,z)\to R_{0j}(\omega\pm i0,z),\quad \varepsilon\to 0+,\quad z\in\mathbb{Z}^3,$$

and

$$\sup_{\varepsilon\in(0,1]}|R_{0j}(\omega\pm i\varepsilon,z)|\le C_j\Big(\ln(1+|z|)+1\Big).$$

Evidently, that the all resolvent R_0 satisfies the similar properties. Hence by the Lebesgue dominated convergence theorem

$$\sum_{x,y\in\mathbb{Z}^3}(1+|x|^2)^{-\sigma}|R_0(\omega\pm i\varepsilon,x-y)-R_0(\omega\pm i0,x-y)|^2(1+|y|^2)^{-\sigma}\to 0,\ \varepsilon\to 0+$$

with $\sigma>3/2$. Then the Hilbert-Schmidt norm of the difference $R_0(\omega\pm i\varepsilon)-R_0(\omega\pm i0)$ converges to zero. Proposition 2.2 is proved. ◻

Remark 2.4. *Differentiating* (2.3) *with respect to* ω*, we obtain similarly that for* $\omega\in\Sigma\setminus\{0,4,8,12\}$ *the derivatives* $\partial_\omega^k R_0(\omega\pm i0)$ *belongs to* $\mathcal{B}(\sigma,-\sigma)$ *with* $\sigma>3/2+k$*.*

2.2 Asymptotics near singular points

We obtain the asymptotics $R_0(\omega)$ near the singular points ω_k. We consider "elliptic" points $\omega_1=0$, $\omega_4=12$ and "hyperbolic" points $\omega_2=4$, $\omega_3=8$ separately.

2.2.1 Asymptotics near elliptic points

Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the point $\omega_1 = 0$ (the expansion near the point $\omega_4 = 12$ can be construct similarly).

Proposition 2.5. *Let $N = 0, 1, 2...$ and $\sigma > N + 3/2$. Then the following expansion holds in $\mathcal{B}(\sigma, -\sigma)$:*

$$R_0(\omega) = \sum_{k=0}^{N} A_{1k}\omega^{k/2} + \mathcal{O}(\omega^{(N+1)/2}),\ |\omega| \to 0,\ \arg\omega \in (0, 2\pi). \tag{2.12}$$

Here $A_{1k} \in \mathcal{B}(\sigma, -\sigma)$ with $\sigma > k + 1/2$.

Proof. The resolvent $R_0(\omega \pm i0)$ is represented by the integral (2.2). Fix $0 < \delta < 1$ and consider $0 < |\omega| < \delta^2/2$. We identify T^3 with the cube $[-\pi, \pi]^3$ and represent $R_0(\omega, z)$, $z = x - y$, as the sum

$$\begin{aligned} R_0(\omega, z) &= \frac{1}{8\pi^3}\int\limits_{T^3} \frac{\chi(|\theta|)e^{-i\theta z}}{\phi(\theta) - \omega}\, d\theta + \frac{1}{8\pi^3}\int\limits_{T^3} \frac{(1-\chi(|\theta|))e^{-i\theta z}}{\phi(\theta) - \omega}\, d\theta \\ &= R_{01}(\omega, z) + R_{02}(\omega, z), \end{aligned} \tag{2.13}$$

where $\chi(r)$ is the smooth function and

$$\chi(r) = \begin{cases} 0, & r > 2\sigma \\ 1, & 0 \le r \le \sigma \end{cases}$$

Since $\phi(\theta) = |\theta|^2 + \mathcal{O}(|\theta|^4)$, then $R_{02}(\omega, z)$ is analytic function of ω in $|\omega| \le \delta^2/2$, and

$$|\partial_\omega^j R_{02}(\omega, z)| \le \frac{C_{j,N}}{(|z|^N + 1)}, \quad |\omega| \le \delta^2/2, \quad z \in \mathbb{Z}^3. \tag{2.14}$$

Hence it suffices to prove the asymptotics of type (2.12) for R_{01}. Let us choose the system of coordinate in which θ_3 coincides with the axe z and rewrite $R_{01}(\omega, z)$ as

$$R_{01}(\omega, z) = \frac{1}{8\pi^3}\int\limits_{|n|=1}\Big(\int\limits_0^{2\delta} \frac{\chi(r)e^{-ir|z|n_3}r^2 dr}{\widetilde{\phi}(rn) - \omega}\Big) ds. \tag{2.15}$$

Here $r = |\theta'|$, $\theta' = rn$, $\widetilde{\phi}(\theta') = \phi(\theta)$. Note that $\widetilde{\phi}$ is even function. Hence,

$$\begin{aligned} R_{01}(\omega, z) &= \frac{1}{8\pi^3}\int\limits_{S_+}\Big(\int\limits_0^{2\delta} \frac{\chi(r)e^{-ir|z|n_3}r^2\, dr}{\widetilde{\phi}(rn) - \omega} + \int\limits_0^{2\delta} \frac{\chi(r)e^{ir|z|n_3}r^2\, dr}{\widetilde{\phi}(rn) - \omega}\Big) ds \\ &= \frac{1}{8\pi^3}\int\limits_{S_+}\Big(\int\limits_{-2\delta}^{2\delta} \frac{\chi(r)e^{-ir|z|n_3}r^2 dr}{\widetilde{\phi}(rn) - \omega}\Big) ds, \end{aligned} \tag{2.16}$$

where $S_+ = \{|n| = 1,\ n_3 > 0$. (We extent $\chi(r)$ on $\mathbb{R}$ as even function). Let us apply the Cauchy residue theorem to the inner integral in (2.16):

$$R_{01}(\omega,z) = \frac{1}{8\pi^3}\int\limits_{S_1^+} \frac{e^{-iq|z|n_3}q^2}{p(\omega,n)}ds + \frac{1}{8\pi^3}\int\limits_{S_+}\Big(\int\limits_{[-2\delta,\delta]\cup[\delta,2\delta]} \frac{\chi(r)e^{-ir|z|n_3}r^2\ dr}{\widetilde{\phi}(rn)-\omega}\Big)ds \tag{2.17}$$
$$+ \frac{1}{8\pi^3}\int\limits_{S_+}\Big(\int\limits_{\Gamma_\delta}\frac{e^{-ir|z|n_3}r^2\ dr}{\widetilde{\phi}(rn)-\omega}\Big)ds = R^1_{01}(\omega,z)+R^2_{01}(\omega,z)+R^3_{01}(\omega,z).$$

Here $q = q(\omega,n)$ is the solution to $\widetilde{\phi}(qn) = \omega$ in the low half-plane $\operatorname{Im} q < 0$, $p(\omega,n) = \partial_r\widetilde{\phi}(rn)\Big|_{r=q}$, $\Gamma_\delta = \{|r| = \delta, \operatorname{Im} r < 0\}$. Since $\widetilde{\phi}(rn) = r^2 + \Phi(rn)$ with $\Phi(rn) = \mathcal{O}(r^4)$, then

$$q(\omega,n) = -\sqrt{\omega}(1+Q(\omega,n)), \quad p(q,n) = \sqrt{\omega}(2+P(\omega,n)), \tag{2.18}$$

where $Q(\omega,n)$, $\Psi(\omega,n)$ are the analytic function of ω, and $Q(\omega,n)$, $\Psi(\omega,n) = \mathcal{O}(\omega)$. Let us consider the third summand in the RHS of (2.17):

$$R^3_{01}(\omega,z) = \frac{1}{8\pi^3}\int\limits_0^{2\pi}d\beta\int\limits_0^{\pi/2}d\alpha\int\limits_{\Gamma_\delta}\frac{e^{-ir|z|\cos\alpha}r^2\sin\alpha\ dr}{\widetilde{\phi}(r,\alpha,\beta)-\omega} \tag{2.19}$$
$$= \frac{1}{8\pi^3|z|}\int\limits_0^{2\pi}d\beta\int\limits_{\Gamma_\delta}dr\int\limits_0^{\pi/2}\frac{irdr}{\widetilde{\phi}(r,\alpha,\beta)-\omega}\ d(e^{-ir|z|\cos\alpha}), \quad |z|\neq 0.$$

Since $|\phi_1(r,\alpha,\beta)-\omega| > C(\delta)\delta$, then the operator value function $R^3_{01}(\omega)$ is analytic and all its derivatives in respect to ω are bounded in $\mathcal{B}(\sigma,-\sigma)$ with $\sigma > 1/2$ for $|\omega| < \delta^2/2$. Hence R^3_{01} admits an expansion of type (2.12).
The expansion of R^2_{01} can been obtained similarly.
Finally, let us consider the first summand in RHS of (2.17):

$$R^1_{02}(\omega,z) = \frac{i}{8\pi^3|z|}\int\limits_0^{2\pi}d\beta\int\limits_0^{\pi/2}\frac{q^2\sin\alpha\ de^{-iq|z|\cos\alpha}}{p(\omega,n)(q\sin\alpha+\partial_\alpha q\cos\alpha)} = \frac{i}{8\pi^3|z|}\Big[\int\limits_0^{2\pi}\frac{q(\omega,\pi/2,\beta)}{p(\omega,\pi/2,\beta)}d\beta$$
$$-\int\limits_0^{2\pi}d\beta\int\limits_0^{\pi/2}e^{-iq|z|\cos\alpha}\ d\frac{q^2\sin\alpha}{p(\omega,n)(q\sin\alpha+\partial_\alpha q\cos\alpha)}\Big] = -\frac{i}{8\pi^3|z|}\Big[\int\limits_0^{2\pi}\frac{1+Q(\omega,\pi/2,\beta)}{2+P(\omega,\pi/2,\beta)}d\beta$$
$$+\int\limits_0^{2\pi}d\beta\int\limits_0^{\pi/2}e^{-i\sqrt{\omega}(1+Q(\omega,\alpha,\beta)|z|\cos\alpha}\ d\frac{(1+Q(\omega,\alpha,\beta))^2}{(2+P(\omega,\alpha,\beta))(1+Q(\omega,\alpha,\beta)+\partial_\alpha Q(\omega,\alpha,\beta)/\tan\alpha)}.$$

Here $q(\omega,n) = q(\omega,\alpha,\beta)$. Expand the integrand, we obtain the asymptotics of type (2.12) for $R^1_{02}(\omega,z)$, since $Q,\ P\sim\omega$. $\square$

Remark 2.6. *The expansion* (2.12) *can been differentiated* $N+1$ *times in* $\mathcal{B}(\sigma,-\sigma)$ *with* $\sigma > N+3/2$*:*

$$\partial^r_\omega R_0(\omega) = \partial^r_\omega\Big(\sum_{k=0}^N A^1_k\omega^{k/2}\Big) + \mathcal{O}(\omega^{(N+1)/2-r}), \quad 1\le r\le N+1.$$

2.2.2 Asymptotics near hyperbolic point

Here we construct the Puiseux expansion of the free resolvent $R_0(\omega)$ near the point $\omega_2 = 4$ (the expansion near the point $\omega_3 = 8$ can be construct similarly). The main contribution into (2.2) is given by the corresponding critical points $(0,0,\pi)$, $(0,\pi,0)$ and $(\pi,0,0)$ of hyperbolic type.

Proposition 2.7. *In $\mathcal{B}(\sigma,-\sigma)$ with $\sigma > 2N + 7/2$ the following expansion holds:*

$$R_0(4+\omega) = \sum_{k=0}^{N} A_{2k}\omega^k + \sqrt{\omega}\sum_{k=0}^{N} B_{2k}\omega^k + \mathcal{O}(\omega^{N+1}),\ |\omega| \to 0,\ \operatorname{Im}\omega > 0, \tag{2.20}$$

where $A_{2k}, B_{2k} \in \mathcal{B}(\sigma,-\sigma)$, wits $\sigma > 2k + 3/2$ are the operators with kernels $A_{2k}(x-y)$, $B_{2k}(x-y)$.

Proof. For $\omega = \omega_2 = 4$ the denominator of the integral (2.2) vanishes along the curve $\phi(\theta) = 4$. We will study main contribution of points $(0,0,\pi)$, $(0,\pi,0)$ and $(\pi,0,0)$ of the curve which are critical points of $\phi(\theta)$. The contribution of other points of the curve can be proved by methods of Section 2.1. For example, let us consider the integral over a neighbourhood of the point $(\pi,0,0)$. Let $\zeta(\theta)$ be a smooth cutoff function, equal 1 in a neighbourhood of the point $(\pi,0,0)$ (the other properties of $\zeta(\theta)$ we specified below). For $\operatorname{Im}\omega > 0$ denote

$$Q(\omega,z) = \frac{1}{8\pi^3}\int \frac{e^{-i(z_1\theta_1+z_2\theta_2+z_3\theta_3)}\ \zeta(\theta)\ d\theta_1\ d\theta_2\ d\theta_3}{\phi(\theta)-4-\omega}$$

$$= \frac{1}{8\pi^3}\int \frac{e^{-i(z_1\theta_1+z_2\theta_2+z_3\theta_3)}\zeta(\theta)\ d\theta_1\ d\theta_2\ d\theta_3}{4\sin^2\frac{\theta_2}{2}+4\sin^2\frac{\theta_3}{2}-4\cos^2\frac{\theta_1}{2}-\omega} = \frac{e^{-iz_1\pi}}{8\pi^3}\int \frac{e^{-i(z_1\theta_1'+z_2\theta_2+z_3\theta_3)}\zeta_1(\theta')\ d\theta_1'\ d\theta_2\ d\theta_3}{4\sin^2\frac{\theta_2}{2}+4\sin^2\frac{\theta_3}{2}-4\sin^2\frac{\theta_1'}{2}-\omega},$$

where $\theta_1' = \theta_1 - \pi$, $\theta' = (\theta_1',\theta_2,\theta_3)$, and $\zeta_1(\theta') = \zeta(\theta)$. We suppose that $\zeta_1(\theta')$ is symmetric in θ_1', θ_2 and θ_3. Then the exponent $e^{-i(z_1\theta_1'+z_2\theta_2+z_3\theta_3)}$ can be substituted by its even part, so we have

$$Q(\omega,z) = \frac{e^{-iz_1\pi}}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\cos(z_1\theta_1')\ \cos(z_2\theta_2)\ \cos(z_3\theta_3)\ \zeta_1(\theta')d\theta_1'd\theta_2d\theta_3}{4\sin^2\frac{\theta_2}{2}+4\sin^2\frac{\theta_3}{2}-4\sin^2\frac{\theta_1'}{2}-\omega} = \frac{e^{-iz_1\pi}}{\pi^3}Q_1(\omega,z), \tag{2.21}$$

Let us obtain the expansion of type (2.20) for Q_1. We change the variables: $s_1 = 2\sin\frac{\theta_2}{2}$, $s_2 = 2\sin\frac{\theta_3}{2}$, and $s_3 = 2\sin\frac{\theta_1'}{2}$ and choose the cutoff function ζ such that $\zeta_1(\theta') = \zeta_2(|s|^2)$, with smooth function ζ_2. Then

$$Q_1(\omega,z) = \int_0^\infty\int_0^\infty\int_0^\infty \frac{F(z,s_1^2,s_2^2,s_3^2)\zeta_2(|s|^2)\ ds_1\ ds_2\ ds_3}{s_1^2+s_2^2-s_3^2-\omega},$$

where

$$F(z,s_1^2,s_2^2,s_3^2) = \cos(z_1 g(s_1))\cos(z_2 g(s_2))\cos(z_3 g(s_3))J(s_1^2)J(s_2^2)J(s_3^2).$$

We change the variables again:

$$s_1 = \tau\cos\varphi, s_2 = \tau\sin\varphi, s_3 = s_3.$$

Then

$$Q_1(\omega,z)=\int\limits_0^\infty\int\limits_0^\infty\frac{F_1(z,\tau^2,s_3^2)\zeta_2(\tau^2+s_3^2)\tau\ d\tau ds_3}{\tau^2-s_3^2-\omega},\tag{2.22}$$

where

$$F_1(z,\tau^2,s_3^2)=\int\limits_0^{\pi/2}F(z,\tau^2\cos^2\varphi,\tau^2\sin^2\varphi,s_3^2)\ d\varphi$$

We change the variables once more:

$$\rho_1=\tau^2-s_3^2=R^2\cos 2\psi,\quad \rho_2=2\tau s_3=R^2\sin 2\psi,\tag{2.23}$$

where R, ψ are the polar coordinates on the plane (τ,s_3). Then $|\rho|^2=\rho_1^2+\rho_2^2=R^4$, hence, $|\rho|=R^2$, and

$$\tau^2=\frac{|\rho|+\rho_1}{2},\qquad s_3^2=\frac{|\rho|-\rho_1}{2}.\tag{2.24}$$

Therefore, $d\rho_1 d\rho_2=4|\rho|d\tau ds_3$, and (2.22) implies

$$Q_1(\omega,z)=\int\limits_0^\infty\Big(\int\limits_{\mathbb{R}}\frac{h(|\rho|,\rho_1,z)}{(\rho_1-\omega)|\rho|}\sqrt{|\rho|+\rho_1}d\rho_1\Big)d\rho_2,\tag{2.25}$$

where $h(|\rho|,\rho_1,z)=F_1(z,\frac{|\rho|+\rho_1}{2},\frac{|\rho|-\rho_1}{2})\zeta_2(|\rho|)/4\sqrt{2}$. Now we can specify all needed properties of cutoff function:

$$\operatorname{supp}\zeta_2(|\rho|)\cap\{\rho\in\mathbb{R}^2:\rho_2\ge 0\}\subset\Pi=\{(\rho_1,\rho_2):-\delta\le\rho_1\le\delta,\ 0\le\rho_2\le\delta\}$$

with some $0<\delta<1$. We consider $0<|\omega|\le\delta/2$, $\operatorname{Im}\omega>0$. Denote $r=r(\rho):=|\rho|$. The function $h(r,\rho_1,z)$ can be expanded into the following finite Taylor series with respect to ρ_1:

$$h(r,\rho_1,z)=h_0(r,z)+h_1(r,z)\rho_1+...+h_N(r,z)\rho_1^N+H_N(r,\rho_1,z)\rho_1^N,\tag{2.26}$$

where $h_k(r,z)$ are polynomial in z of order $2k$, and

$$|H_N(r,\rho_1,z)|\le C|z|^{2N},\quad |\partial_{\rho_1}H_N(r,\rho_1,z)|\le C|z|^{2N+2},\ (\rho_1,\rho_2)\in[-\delta,\delta]\times[0,\delta],\ r=|\rho|.\tag{2.27}$$

Step i) Let us substitute (2.26) into (2.25) and consider the summands with $h_k(r,z)$, $k=0,1,...,N$:

$$\begin{aligned}\int\limits_\Pi\frac{h_k(r,z)\rho_1^k\sqrt{r+\rho_1}}{(\rho_1-\omega)r}d\rho_1 d\rho_2&=\int\limits_\Pi\frac{h_k(r,z)\sqrt{r+\rho_1}}{r}\Big(\rho_1^{k-1}+\omega\rho_1^{k-2}+...+\omega^{k-1}+\frac{\omega^k}{\rho_1-\omega}\Big)d\rho_1 d\rho_2\\&=\sum_{k=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_\Pi\frac{h_k(r,z)\sqrt{r+\rho_1}}{(\rho_1-\omega)r}d\rho_1 d\rho_2\\&=\sum_{j=0}^{k-1}a_{k,j}(z)\omega^j+\omega^k\int\limits_0^\delta\frac{h_k(r,z)}{\sqrt{r}}dr\int\limits_0^\pi\frac{\sqrt{1+\cos\psi}\ d\psi}{\cos\psi-\omega/r},\end{aligned}\tag{2.28}$$

where $a_{k,j}(z)$ are polynomial in z of order $2k$. Let us calculate the integral

$$\int\limits_0^\pi \frac{\sqrt{1+\cos\psi}\ d\psi}{\cos\psi-\omega/r} = \int\limits_0^\pi \frac{2\sqrt{2}d(\sin\frac{\psi}{2})}{1-2\sin^2\frac{\psi}{2}-\omega/r} = -\int\limits_0^1 \frac{\sqrt{2}dt}{t^2-\frac{r-\omega}{2r}}$$
$$= -\frac{\sqrt{r}}{\sqrt{r-\omega}}\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}+\frac{\pi i\sqrt{r}}{\sqrt{r-\omega}}, \quad \operatorname{Im}\omega>0. \tag{2.29}$$

Here $\sqrt{r}\geq 0$. Function $z=\sqrt{r-\omega}$ is analytic in $\operatorname{Im}\omega>0$ with the values in $\operatorname{Im}z<0$, $\operatorname{Re}z>0$, and function $\zeta=\log w$ is analytic in $|w|<1$, $\operatorname{Im}w>0$, where $\log(-1)=\pi i$.

Substitute (2.29) into (2.28), we get

$$\int\limits_\Pi \frac{h_k(r,z)\rho_1^k}{(\rho_1-\omega)r}\sqrt{r+\rho_1}d\rho_1 d\rho_2 = \sum_{j=0}^{k-1} a_{k,j}(z)\omega^j + \omega^k\int\limits_0^\delta \left(\pi i - \log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{h_k(r,z)dr}{\sqrt{r-\omega}} \tag{2.30}$$

Let us expand $h_k(r,z)$ into the following finite Taylor series with respect to r:

$$h_k(r,z)=h_{k,0}(z)+h_{k,1}(z)r+...+h_{k,N-k}(z)r^{N-k}+H_{k,N-k}(r,z)r^{N-k}, \tag{2.31}$$

where $h_{k,j}(z)$ are polynomial in z of order $2(k+j)$, and $|H_{k,N-k}(r,z)|\leq C|z|^{2N}$, $0\leq r\leq\delta$. The following lemma is true

Lemma 2.8. *Let* $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$. *Then*

$$I_l=\int_0^\delta \left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{r^l dr}{\sqrt{r-\omega}} = s_l(\omega)+C_l\omega^l\sqrt{\omega}, \tag{2.32}$$

where s_l *are analytic in* $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$, $C_l\in\mathbb{R}$.

We shall prove this lemma in Appendix B. Now (2.30), (2.31) and (2.32) imply that for $0<|\omega|<\delta/2$, $\operatorname{Im}\omega>0$

$$\int\limits_\Pi \frac{h_k(r,z)\rho_1^k}{(\rho_1-\omega)r}\sqrt{r+\rho_1}d\rho_1 d\rho_2 = \sum_{j=0}^{N}\tilde{a}_{k,j}(z)\omega^j+\omega^k\sqrt{\omega}\sum_{j=0}^{N-k}b_{k,j}(z)\omega^j+\hat{a}_{N+1,k}(\omega,z)\omega^{N+1}$$
$$+\ \omega^k\int\limits_0^\delta\left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{H_{k,N-k}(r,z)r^{N-k}dr}{\sqrt{r-\omega}}, \tag{2.33}$$

where $|\tilde{a}_{k,j}(z)|\leq C(k,j)|z|^{2N}$, $|b_{k,j}(z)|\leq C(k,j)|z|^{2(k+j)}$ and $|\hat{a}_{N+1,k}(\omega,z)|\leq C(N+1,k)|z|^{2N}$.

Let us consider the integral in RHS of (2.33) for $0<|\omega|<\delta/2$:

$$\int\limits_0^\delta\left(\pi i-\log\frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\frac{H_{k,N-k}(r,z)r^{N-k}dr}{\sqrt{r-\omega}} = \int\limits_0^{2|\omega|}+\int\limits_{2|\omega|}^\delta = I_1+I_2. \tag{2.34}$$

In the first summand we change the variable: $r = |\omega|\tau$. Then

$$\begin{aligned} |I_1| &= |\omega|^{N-k}\sqrt{|\omega|}\,\Big|\int\limits_0^2 \left(\pi i - \log\frac{1-\sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}}{1+\sqrt{\frac{\tau-\omega/|\omega|}{2\tau}}}\right)\frac{H_{k,N-k}(|\omega|\tau, z)\tau^{N-k}d\tau}{\sqrt{\tau-\omega/|\omega|}}\Big| \qquad (2.35)\\ &\le C|z|^{2N}|\omega|^{N-k}\sqrt{|\omega|}, \end{aligned}$$

Let us consider the second summand in RHS of (2.34):

$$\begin{aligned} I_2 &= \int\limits_{2|\omega|}^{\delta} H_{k,N-k}(r,z)r^{N-k-1/2}\Big(d_0 + d_1\frac{\omega}{r} + ... + d_{N-k}\frac{\omega^{N-k}}{r^{N-k}} + \hat{d}_{N-k}(\omega/r)\frac{\omega^{N-k}}{r^{N-k}}\Big)dr \\ &= \int\limits_{2|\omega|}^{\delta} H_{k,N-k}(r,z)\Big(d_0 r^{N-k-1/2} + d_1\omega r^{N-k-3/2} + ... + d_{N-k}\omega^{N-k}r^{-1/2}\Big)dr + \tilde{u}_{N-k}(\omega,z) \\ &= \int\limits_{0}^{\delta} H_{k,N-k}(r,z)\Big(d_0 r^{N-k-1/2} + d_1\omega r^{N-k-3/2} + ... + d_{N-k}\omega^{N-k}r^{-1/2}\Big)dr + \hat{u}_{N-k}(\omega,z) \\ &= \sum_{j=0}^{N-k} u_j(z)\omega^j + \hat{u}_{N-k}(\omega,z), \qquad (2.36) \end{aligned}$$

where $|u_j(z)| \le C|z|^{2N}|$; $|\tilde{u}_{N-k}(\omega,z)|$, $|\hat{u}_{N-k}(\omega,z)| \le C|z|^{2N}|\omega|^{N-k}$.
Step ii) It remains to consider the contribution into (2.25) from the remainder $H_N(r,\rho_1,z)\rho_1^N$:

$$\begin{aligned} &\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\rho_1^N d\rho_1 d\rho_2}{(\rho_1-\omega)r}\sqrt{r+\rho_1} \\ &= \int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}}{r}\Big(\rho_1^{N-1} + \omega\rho_1^{N-2} + ... + \omega^{N-1} + \frac{\omega^N}{\rho_1-\omega}\Big)d\rho_1 d\rho_2 \\ &= \sum_{j=0}^{N-1} w_j(z)\omega^j + \omega^N\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\;d\rho_1 d\rho_2}{(\rho_1-\omega)r}, \qquad (2.37) \end{aligned}$$

where $|w_j(z)| \le C|z|^{2N}$. For the integral in RHS of (2.37) the following estimate is true

Lemma 2.9. *Let* $0 < |\omega| < \delta/2$, $\operatorname{Im}\omega > 0$. *Then*

$$\Big|\int\limits_{\Pi}\frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\;d\rho_1 d\rho_2}{(\rho_1-\omega)r}\Big| \le C|z|^{2N}\ln^2|z|,\ |z|>1. \qquad (2.38)$$

We shall prove this lemma in Appendix C.
Step iii) Finally, we obtain

$$Q_1(\omega,z) = \sum_{k=0}^{N} q_k(z)\omega^k + \sqrt{\omega}\sum_{k=0}^{N} p_k(z)\omega^k + \widehat{q}_N(\omega,z),\ |\omega|\to 0,$$

where $|\widehat{q}_N(\omega,z)| \le C|z|^{2N}\ln^2|z||\omega|^N$. Further, $p_k(z) = \mathcal{O}(|z|^{2k})$, and $q_k(z) = \mathcal{O}(|z|^{2N})$ for $0 \le k \le N$. Therefore, $q_k(z) = \mathcal{O}(|z|^{2k})$, since $q_k(z)$ do not depend on N. $\square$

Remark 2.10. *The expansions* (2.20) *can be differentiated* $2N+2$ *times in* $\mathcal{B}(\sigma,-\sigma)$ *with* $\sigma > 2N+7/2$. *More precisely,*

$$\partial_\omega^r R_0(4+\omega) = \partial_\omega^r\Big(\sum_{k=0}^N A_{2k}\omega^k + \sqrt{\omega}\sum_{k=0}^N B_{2k}\omega^k\Big) + \mathcal{O}(\omega^{N+1-r}), \quad 1 \le r \le 2N+2$$

3 Perturbed resolvent

3.1 The limiting absorption principle

In the next proposition we develop the results of [3], [14] for the 3D case and prove the limiting absorption principle in the sense of the operator convergence. It will be needed for the proof of the long-time asymptotics (1.6).

Proposition 3.1. *Let* $V \in \mathcal{V}$, $\sigma > 3/2$. *Then the following limits exist as* $\varepsilon \to 0+$

$$R(\omega \pm i\varepsilon) \xrightarrow{B(\sigma,-\sigma)} R(\omega \pm i0), \quad \omega \in \Sigma \setminus \{0,4,8,12\}. \tag{3.1}$$

Proof. The existence of the limit (3.1) follow from the representation $R = R_0(I+VR_0)^{-1}$, Proposition 2.2 and from the invertibility of the operators $I+VR_0(\omega \pm i0$. The invertibility of the operators $I+VR_0(\omega \pm i0$ follow from [14, Theorem 9]. □

Remark 3.2. *For* $\omega \in \Sigma \setminus \{0,4,8,12\}$ *the derivatives* $\partial_\omega^k R(\omega \pm i0)$ *belong* $\mathcal{B}(\sigma,-\sigma)$ *with* $\sigma > 3/2+k$.

3.2 Asymptotics near singular points

In this sections we are going to obtain an asymptotic expansion for the perturbed resolvent $R(\omega \pm i0)$ near $\omega = 0,4,8,12$.

Definition 3.3. *i) A set* $\mathcal{W} \subset \mathcal{V}$ *is called generic, if for each* $V \in \mathcal{V}$ *we have* $\alpha V \in \mathcal{W}$, *with the possible exception of a discrete set of* $\alpha \in \mathbb{R}$.
ii) We say that a property holds for a "generic" V, *if it holds for all* V *from a generic subset of* $\mathcal{V}$.

Theorem 3.4. *Let* $\sigma > 3/2$. *Then for "generic"* $V \in \mathcal{V}$ *the following expansion holds:*

$$R(\omega) = D_1 + \mathcal{O}(\sqrt{\omega}), \qquad |\omega| \to 0, \ \arg\omega \in (0,2\pi) \tag{3.2}$$

in the norm of $\mathcal{B}(\sigma,-\sigma)$, *where* D_1 *is the operator with the kernel* $D_1(x,y)$.

Proof. We use the relation

$$R(\omega) = T^{-1}(\omega)R_0(\omega), \quad \text{where} \quad T(\omega) := I + R_0(\omega)V. \tag{3.3}$$

According to (2.12),

$$T(\omega) = I + A_{10}V + O(\sqrt{\omega}), \qquad |\omega| \to 0, \quad \arg\omega \in (0,2\pi). \tag{3.4}$$

Let us prove that for "generic" $V \in \mathcal{V}$ the operator $T(\omega)$ is invertible in $l^2_{-\sigma}$ for sufficient small $|\omega| > 0$. It is suffices to prove that for "generic" $V \in \mathcal{V}$ the operator $T(0) = \mathrm{Op}\Big[\delta(x-y) + A_{10}V(y)\Big]$ is invertible in $l^2_{-\sigma}$, or the operator

$$\mathrm{Op}\Big[(1+x^2)^{-\sigma/2}(\delta(x-y) + A_{10}V(y))(1+y^2)^{\sigma/2}\Big]$$

is invertible in l^2. Let us consider the operator

$$\mathcal{A}(\alpha) = \mathrm{Op}[(1+x^2)^{-\sigma/2}\big(\delta(x-y) + \alpha A_{10}V(y)\big)(1+y^2)^{\sigma/2}] = 1 + \alpha\mathcal{K}, \qquad \alpha \in \mathbb{C}.$$

For $\sigma > 3/2$

$$K(x,y) = (1+x^2)^{-\sigma/2}A_{10}V(y)(1+y^2)^{\sigma/2} \in l^2(\mathbb{Z}^2 \times \mathbb{Z}^2).$$

Hence, $K(x,y)$ is a Hilbert-Schmidt kernel, and accordingly the operator $\mathcal{K} = \mathrm{Op}(K(x,y))$: $l^2 \to l^2$ is compact. Further, $\mathcal{A}(\alpha)$ is analytic in $\alpha \in \mathbb{C}$, and $\mathcal{A}(0)$ is invertible. It follows that $\mathcal{A}(\alpha)$ is invertible for all $\alpha \in \mathbb{R}$ outside a discrete set; see [2]. Thus we could replace the original potential V by αV with α arbitrarily close to 1, if necessary, to have $T_r(0)$ invertible.

Now (3.3) and (3.4) imply that for sufficiently small $|\omega| > 0$

$$R(\omega) = (I + T(0) + \mathcal{O}(\sqrt{\omega}))^{-1}(A_{10} + \mathcal{O}(\sqrt{\omega})) = T(0)^{-1}A_{10} + O(\sqrt{\omega}). \tag{3.5}$$

$\square$

Remark 3.5. *i) The expansion of resolvent near the second elliptic point $\omega_4 = 12$ is similar to the expansion (3.2).*
ii) The expansion of type (3.2) near the hyperbolic points $\omega_2 = 4$ and $\omega_3 = 8$ require larger value of σ. Namely, for "generic" $V \in \mathcal{V}$

$$R(\omega_k + \omega) = D_k + \mathcal{O}(\sqrt{\omega}), \quad |\omega| \to 0, \quad \mathrm{Im}\,\omega > 0$$

in the norm of $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 7/2$.
iii) These expansion can be differentiated two times in $\mathcal{B}(\sigma, -\sigma)$ with $\sigma > 5/2$ for elliptic points and with $\sigma > 7/2$ for hyperbolic points. In these cases $\partial^2_\omega R(\omega_k + \omega) = \mathcal{O}(\omega^{-3/2})$, $k = 1, 2, 3, 4$.

4 Long-time asymptotics

Theorem 4.1. *Let $\sigma > 7/2$. Then for "generic" $V \in \mathcal{V}$ the following asymptotics hold*

$$\Big\|e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j}P_j\Big\|_{B(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \ t \to \infty. \tag{4.1}$$

Here P_j denote the projections on the eigenspaces corresponding to the eigenvalues $\mu_j \in \mathbb{R} \setminus [0, 12]$, $j = 1, \ldots, n$.

Proof. The estimate (4.1) is based on the formula

$$e^{-itH} = -\frac{1}{2\pi i} \oint\limits_{|\omega|=C} e^{-it\omega} R(\omega) d\omega, \ C > \max\{12; |\mu_j|, \ j = 1, ..., n\}. \tag{4.2}$$

Therefore

$$e^{-itH} - \sum_{j=1}^{n} e^{-it\mu_j} P_j = \frac{1}{2\pi i} \int\limits_{[0,12]} e^{-it\omega} (R(\omega + i0) - R(\omega - i0)) \, d\omega = \int\limits_{[0,12]} e^{-it\omega} P(\omega) d\omega,$$

The main contribution into the long-time asympotics gives the integrals over the neighbourhoods of the singular points $\omega = 0, 4, 8, 12$. For example, let us consider the integral over the neighbourhood of the point $\omega_1 = 0$. The expansion (3.2) and Remark 3.5 imply

$$P^{(k)}(\omega) = \mathcal{O}(\partial^k \sqrt{\omega}), \ \omega \to +0, \ \omega \in \mathbb{R}, \ k = 0, 1, 2 \tag{4.3}$$

in the norm of $B(\sigma, -\sigma)$ with $\sigma > 7/2$. Then by Lemma 10.2 from [7] we get

$$\int\limits_{0}^{1} e^{-it\omega} \zeta(\omega) P(\omega) = \mathcal{O}(t^{-3/2}), \quad t \to \infty,$$

in the norm of $B(\sigma, -\sigma)$ with $\sigma > 7/2$. Where $\zeta(\omega)$ is the smooth function, $\operatorname{supp} \zeta \in (-1, 1)$, and $\zeta(\omega) = 1$ for $\omega \in [0, 1/2]$. The integrals over the neighbourhoods of the points $\omega = 4, 8, 12$ can be estimated similarly. □

5 Application to the asymptotic completeness

We apply the obtained results to prove the asymptotic completeness which follows by standard Cook's argument.

Theorem 5.1. *i) Let $\psi_0 \in l^2$. Then for "generic" $V \in \mathcal{V}$ for solution to* (1.1) *the following long time asymptotics hold*

$$\psi(\cdot, t) = \sum_{j=1}^{n} \sum_{k_j} C_{k_j} e^{-it\mu_j} u_{k_j} + U_0(t) \phi_{\pm} + r_{\pm}(t), \quad t \to \pm\infty, \tag{5.4}$$

where u_{k_j} are the corresponding eigenfunctions, $\phi_{\pm} \in l^2$ are the scattering states, and

$$\|r_{\pm}(t)\|_{l^2} \to 0, \quad t \to \pm\infty.$$

ii) Furthermore,

$$\|r_{\pm}(t)\|_{l^2} = \mathcal{O}(|t|^{-1/2}) \tag{5.5}$$

if $\psi_0 \in l^2_\sigma$ with $\sigma > 7/2$.

Proof. We will consider the asymptotics for $t \to \infty$. It suffices to prove the asymptotics (5.4) for $\psi_0 \in l^2_\sigma$ with $\sigma > 7/2$ since l^2_σ is dense in l^2 and the group U_0 is unitary in l^2. Let us apply the projector P onto the continuous spectrum to both sides of (1.1):

$$iP\dot{\psi} = H_0 P\psi + VP\psi. \tag{5.6}$$

Hence, the Duhamel representation gives,

$$P\psi(t) = U_0(t)P\psi(0) + \int_0^t U_0(t-\tau)VP\psi(\tau)d\tau, \qquad t \in \mathbb{R}. \tag{5.7}$$

We can rewrite (5.7) as

$$P\psi(t) = U_0(t)\Big(P\psi(0) + \int_0^\infty U_0(-\tau)VP\psi(\tau)d\tau\Big) - \int_t^\infty U_0(t-\tau)VP\psi(\tau)d\tau = U_0(t)\phi + r_+(t). \tag{5.8}$$

Now we can prove that the first integral in (5.8) converges, and the function $\int_0^\infty U_0(-\tau)VP\psi(\tau)d\tau$ belongs to l^2. Indeed, the unitarity of $U_0(t)$ in l^2 implies by (4.1) that

$$\begin{aligned}\int_0^\infty \|U_0(-\tau)VP\psi(\tau)\|_{l^2}d\tau &= \int_0^\infty \|VP\psi(\tau)\|_{l^2}d\tau \le C\int_0^\infty \|P\psi(\tau)\|_{l^2_{-\sigma}}d\tau \\ &\le C\int_0^\infty t^{-3/2}\|\psi(0)\|_{l^2_\sigma}d\tau \le C\end{aligned}$$

Here we used that P is bounded operator in $l^2_{-\sigma}$ since $P = I - \sum_{j=1}^n P_j$. The estimate (5.5) for $r_+(t)$ follows similarly. $\square$

6 The Klein-Gordon equation

Now we extend the results of Sections 3-4 to the case of the Klein-Gordon equation (1.8). The resolvent $\mathbf{R}(\omega)$ can be expressed in term of the resolvent $R(\omega)$ as

$$\mathbf{R}(\omega) = \left(\begin{array}{cc} \omega R(\omega^2 - m^2) & iR(\omega^2 - m^2) \\ -i(1 + \omega^2 R(\omega^2 - m^2)) & \omega R(\omega^2 - m^2)\end{array}\right). \tag{6.9}$$

The representation (6.9) implies the following long time asymptotics:
Let $\sigma > 7/2$. Then for "generic" $V \in \mathcal{V}$

$$\Big\| e^{-it\mathbf{H}} - \sum_\pm \sum_{j=1}^n e^{-it\nu_j^\pm}\mathbf{P}_j^\pm \Big\|_{\mathbf{B}(\sigma,-\sigma)} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.$$

Here $\mathbf{P}_j^\pm$ are the projections onto the eigenspaces corresponding to the eigenvalues $\nu_j^\pm = \pm\sqrt{m^2 + \omega_j}$, $j = 1, \dots, n$.

7 Appendix A

Let us prove Lemma 2.8. For $l = 0$ we get

$$I_0 = \int_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{dr}{\sqrt{r-\omega}} = 2\sqrt{r-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right)\Big|_0^\delta$$

$$+ 2\sqrt{2}\omega \int_0^\delta \frac{dr}{\sqrt{r}(r+\omega)} = 2\sqrt{\delta-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{\delta-\omega}{2\delta}}}{1+\sqrt{\frac{\delta-\omega}{2\delta}}}\right) - i2\sqrt{2\omega}\log\frac{\sqrt{r}-i\sqrt{\omega}}{\sqrt{r}+i\sqrt{\omega}}\Big|_0^\delta$$

$$= \tilde{s}_0(\omega) - i2\sqrt{2\omega}\log\frac{1-i\sqrt{\frac{\omega}{\delta}}}{1+i\sqrt{\frac{\omega}{\delta}}} - \pi\sqrt{2\omega} = s_0(\omega) + C_0\sqrt{\omega},$$

where $\tilde{s}_0$, s_0 are the analytic functions of ω, $C_0 = -\pi\sqrt{2}$. Further, for $l \geq 1$ we get

$$I_l = \int_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{r^l dr}{\sqrt{r-\omega}} = 2\delta^l\sqrt{\delta-\omega}\left(\pi i - \log \frac{1-\sqrt{\frac{\delta-\omega}{2\delta}}}{1+\sqrt{\frac{\delta-\omega}{2\delta}}}\right)$$

$$- \quad 2l\int_0^\delta \frac{r^{l-1}(r-\omega)}{\sqrt{r-\omega}}\left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) dr + 2\sqrt{2}\omega\int_0^\delta \frac{r^l dr}{\sqrt{r}(r+\omega)}$$

$$= \quad \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} + 2\sqrt{2}\omega\int_0^\delta \frac{dr}{\sqrt{r}}\left(r^{l-1} - \omega r^{l-2} + \ldots + (-\omega)^{l-1} + \frac{(-\omega)^l}{r+\omega}\right)$$

$$= \quad \tilde{s}_l(\omega) - 2lI_l + 2l\omega I_{l-1} + \tilde{\tilde{s}}_l(\omega) - i2\sqrt{2\omega}(-\omega)^l\log\frac{1-i\sqrt{\frac{\omega}{\delta}}}{1+i\sqrt{\frac{\omega}{\delta}}} - 2\pi\sqrt{2\omega}(-\omega)^l,$$

where $\tilde{s}_l$, $\tilde{\tilde{s}}_l$ are the analytic functions of ω. Hence, $I_l = s_l(\omega) + C_l\omega^l\sqrt{\omega}$, where s_l is the analytic functions of ω and $C_l \in \mathbb{R}$.

8 Appendix B

Here we prove Lemma 2.9. We estimate only integral over $\Pi_+ = \{0 \leq \rho_1, \rho_2 \leq \delta\}$. The integral over $\Pi \setminus \Pi_+$ can been estimated similarly. Let us split the integral over Π_+ into two integrals:

$$\int_{\Pi_+} \frac{H_N(r,\rho_1,z)\sqrt{r+\rho_1}\; d\rho_1 d\rho_2}{(\rho_1-\omega)r} = \int_{\Pi_+} \frac{(H_N(r,\rho_1,z) - H_N(r,|\omega|,z))\sqrt{r+\rho_1}d\rho_1 d\rho_2}{(\rho_1-\omega)r}$$

$$+ \int_{\Pi_+} \frac{H_N(r,|\omega|,z)\sqrt{r+\rho_1}d\rho_1 d\rho_2}{(\rho_1-\omega)r} = J_1 + J_2.$$

Similar (2.29) we obtain

$$J_2 = \int_0^\delta \left(\pi i - \log \frac{1-\sqrt{\frac{r-\omega}{2r}}}{1+\sqrt{\frac{r-\omega}{2r}}}\right) \frac{H_N(r,|\omega|,z)dr}{\sqrt{2(r-\omega)}}.$$

Note that

$$\Big|\log\Big|\frac{\sqrt{2r}-\sqrt{r-\omega}}{\sqrt{2r}+\sqrt{r-\omega}}\Big|\Big| = \Big|\log\Big|\frac{r+\omega}{(\sqrt{2r}+\sqrt{r-\omega})^2}\Big|\Big|$$

$$\le |\log|r+\omega|| + 2|\log|\sqrt{2r}+\sqrt{r-\omega}|| \le 2|\log|r-|\omega|||.$$

Then (2.8) implies

$$|J_2| \le C|z|^{2N}\int\limits_0^\delta \frac{1+|\log|r-|\omega|||}{|\sqrt{r-|\omega|}|}dr \le C|z|^{2N}.$$

Further, for $|z|>1$ let us split J_1 as

$$J_1 = J_{11} + J_{12} + J_{13},$$

where J_{11} is integral over $\Pi_1 = \{(\rho_1,\rho_2)\in\Pi_+ : |r| < 1/|z|^{4/3}\}$, J_{12} is integral over $\Pi_2 = \{(\rho_1,\rho_2)\in\Pi_+\setminus\Pi_1 : |\rho_1-|\omega|| < 1/|z|^{8/3}\}$, and J_{13} is integral over $\Pi_3 = \Pi_+\setminus(\Pi_1\cup\Pi_2)$ (see picture 1).

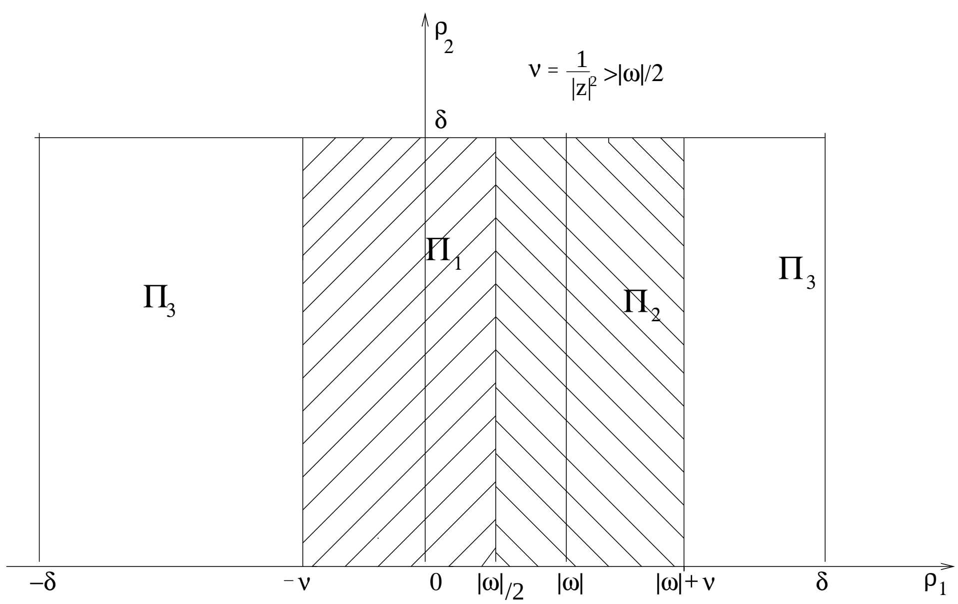

Figure 1: $|\omega| - 1/|z|^{8/3} < 0$.

By (2.27) and the inequality $|\rho_1-\omega| \ge |\rho_1-|\omega||$ we get

$$|J_{11}| \le C|z|^{2N+2}\int\limits_{\Pi_1}\frac{|\rho_1-|\omega||\sqrt{r+\rho_1}d\rho_1 d\rho_2}{|\rho_1-\omega|r} \le C|z|^{2N+2}\int\limits_{\Pi_1}\frac{\sqrt{r+\rho_1}d\rho_1 d\rho_2}{r}$$

$$\le C|z|^{2N+2}\int\limits_0^{\pi/2}d\psi\int\limits_0^{1/|z|^{4/3}}\sqrt{r+r\cos\psi}\;dr \le C|z|^{2N+2}\int\limits_0^{\pi/2}\cos\frac{\psi}{2}d\psi\int\limits_0^{1/|z|^{4/3}}\sqrt{r}dr \le C|z|^{2N}.$$

For the second integral we obtain similarly

$$|J_{12}| \le C|z|^{2N+2}\int\limits_{\Pi_2}\frac{\sqrt{r+\rho_1}d\rho_1 d\rho_2}{r} \le C|z|^{2N+2}\int\limits_{\Pi_2}\frac{d\rho_1 d\rho_2}{\sqrt{r}} \le C|z|^{2N+2}|z|^{2/3}\frac{\delta}{|z|^{8/3}} \le C|z|^{2N},$$

since $1/\sqrt{r} \le |z|^{2/3}$ for $(\rho_1,\rho_2)\in\Pi_2$, and $|\Pi_2| \le 2\delta/|z|^{8/3}$. Finally, (2.27) implies

$$|J_{13}| \le C|z|^{2N}\int\limits_{\Pi_3}\frac{d\rho_1 d\rho_2}{|\rho_1-|\omega||\sqrt{\rho_1^2+\rho_2^2}} \le C|z|^{2N}\ln^2|z|,$$

since for any vertical interval $I\in\Pi_3$ we get

$$\int\limits_I \frac{d\rho_2}{\sqrt{\rho_1^2+\rho_2^2}} = \ln(\rho_2+\sqrt{\rho_1^2+\rho_2^2}) \le C\ln|z|.$$

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